ON PLANE ALGEBRAIC CURVES SYMMETRICAL WITH RESPECT TO EACH OF TWO RECTANGULAR AXES.*

By PROFESSOR R. D. CARMICHAEL, Anniston, Alabama.

The object of this paper is to point out the form of the Cartesian equation of plane algebraic curves symmetrical with respect to each of two rectangular axes, and to classify such curves of the fourth degree—the grouping in classes being determined by certain geometric properties common to those of each class.

The axes of symmetry will be taken as the axes of coordinates. Then if one point is (α, β), three other points are evidently $(-\alpha, \beta)$, $(-\alpha, -\beta)$, $(\alpha, -\beta)$. Since to each value α of x there correspond two values $+\beta$ and $-\beta$ of y, it follows that y enters into the equation only in even powers. In the same manner it may be shown that x enters only in even powers. Therefore,

If a plane algebraic curve symmetrical with respect to each of two rectangular axes is referred to these axes as axes of coordinates, its equation has only terms of even degree in both x and y. The curve is, therefore, of even order. These conditions are evidently sufficient, as well as necessary, for the existence of the defined symmetry; for if x and y enter to only even degrees, to each point (α, β) of the locus will correspond three others $(-\alpha, \beta)$, $(-\alpha, -\beta)$, $(\alpha, -\beta)$ — a condition which is clearly sufficient for the existence of the symmetry in consideration.

This result indicates, as it should, that the circle, ellipse, and hyperbola all possess such symmetry while the parabola does not.

The classification of quartic curves possessing such symmetry is not so simple a matter. For the resolution of this question will be required certain of Plücker's equations. By n, m. δ, τ, ρ, ι, we shall as usual represent respectively, the order, class, number of double points, number of double tangents, number of cusps, number of points of inflection of the curve. The Plücker equations which we shall require are then the following:

$$m=n(n-1)-(2\delta+3\rho), \tag{1}$$
$$n=m(m-1)-(2\tau+3\iota), \tag{2}$$
$$\iota=3n(n-2)-(6\delta+8\rho), \tag{3}$$
$$\rho=3m(m-2)-(6\tau+8\iota). \tag{4}$$

We have now to find the cases in which these equations can be satisfied subject to the condition that the defined symmetry exists.

Evidently, the singularities which are not on the axes can enter only by fours; and to each singularity on one axis and not at the origin must correspond another on the same axis and on the opposite side of the origin.

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Hence, singularities on an axis and not at the origin must enter by pairs. Moreover, it is easy to see that cusps and points of inflection can enter even at the origin only in pairs. Again, if a point of inflection occurs anywhere on an axis there will be two coincident points of inflection, as may be seen by a consideration of the geometric nature of a point of inflection and of the curves having the defined symmetry. To each of these two will correspond another on the opposite side of the origin. Hence, points of inflection not at the origin can enter only by fours. Again: if a double point occurs on an axis, there will be two coincident double points unless the two branches meeting in the point osculate and have the axis as a common tangent. If such an osculating point exists on one side of the origin it exists also on the axis on the other side. Therefore, since each point of tangency counts as two points of intersection, the axis will intersect the curve in eight points —a condition which is impossible for quartic and sextic curves. Therefore, in quartic and sextic curves double points not at the origin enter only by fours. In the same way it may be shown that in these quartic and sextic curves cusps not at the origin can enter only by fours.

Let us now apply the results of the foregoing paragraph to quartic curves having the defined symmetry. We have seen that ρ and ι are even. But by (3), $\rho \ngtr 3$ and $\iota \ngtr 24$. Hence $\rho=0$ or 2. How, if $\rho=2$, both cusps must be at the origin; for we have seen that cusps not at the origin can enter only by fours. If two cusps exist at the same point there will be a double point. But with $\rho=2$ and δ not zero, equations (1) to (4) can exist at the same time only when $\rho=2$, $\delta=1$, $\tau=1$, $\iota=2$. In the present case the two points of inflection can exist only at the origin; for such points not at the origin can enter only by fours. This introduces a second double point at the origin contrary to the equation $\delta=1$. Hence $\rho \neq 2$. Then $\rho=0$. With $\rho=0$, only the following sets of values will satisfy equations (1) to (4):

	I.	II.	III.	IV.
$\rho=$	0,	0,	0,	0.
$\delta=$	3,	2,	1,	0.
$\tau=$	4,	8,	16,	28.
$\iota=$	6,	12,	18,	24.
$m=$	6,	8,	10,	12.

These are the only four possible cases for quartic curves. Moreover, since double points not at the origin can enter only by fours, it follows that in each of the first three cases, the curves must pass through the origin; and therefore for these cases the independent term in the equation is always lacking.

It will be observed that in order to determine to which class any given curve belongs we have only to inquire how many double points it has at the origin. The first case, $\delta=3$, can exist only when the origin alone is the locus

of the equation; for otherwise some line through the origin would cut the curve in five points, which is impossible.

We may also apply certain of the foregoing results to sextic curves whose equation is of the form

$$a_0x^6+a_1x^4y^2+a_2x^2y^4+a_3y^6+a_4x^4+a_5x^2y^2 \\ +a_6y^4+a_7x^2+a_8y^2+a_9=0, \quad a_9\neq 0.$$

Since this curve does not pass through the origin, its double points, points of inflection, and cusps can enter only by fours. Then from (1) it follows that $\rho=0$, 4, or 8; and $\delta=0$, 4, 8, or 12. Also, from (2) $m \nless 3$. Therefore, equations (1) to (4) yield only the following sets of values:

	I.	II.	III.	IV.	V.	VI.	VII.
$\rho=$	0,	0,	0,	0,	4,	4,	8.
$\delta=$	0,	4,	8,	12,	0,	4,	0.
$\iota=$	72,	48,	24,	0,	40,	16,	8.
$\tau=$	324,	156,	52,	12,	40,	36,	0.
$m=$	30,	22,	14,	6,	18,	10,	6.

ON THE DETERMINATION OF CONICS THROUGH TWO POINTS, THE MAJOR AXIS AND ONE FOCUS BEING GIVEN.

By T. H. HILDEBRANDT, The University of Chicago.

In the consideration of the question of the determination of constants in the application of the Principle of Least Action to Planetary Motion, we need to solve the following problem:

"Given two points, a focus, and a major axis, of a conic, besides its nature (*i. e.*, ellipse, hyperbola, or parabola). Required the number of conics that will satisfy the given conditions." This problem has been solved for the case of the ellipse by Jacobi.* The regions in which the second point must lie in order that we obtain a real solution, have also been determined.† So far as I know, however, the problem has not been considered for the cases in which the conic is an hyperbola or a parabola.

I. *The Hyperbola.* We need to use the following property of the hyperbola: The difference of the lengths of the focal radii of an hyperbola is equal in length to the major axis.

Let S be one focus and F another, and let P be a point on the hyper-

*Jacobi, *Vorlesungen ueber die Dynamik.* Werke, Suppl., p. 48.

†Todhunter, *Researches in Calculus of Variations*, p. 162.